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Citation style: Łydzińska Grażyna. (2015). The characterization of some set-valued iteration semigroups. "Aequationes Mathematicae" (Vol. 89, no. 3 (2015), s. 791-802), doi 10.1007/s00010-014-0270-x

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Aequat. Math. 89 (2015), 791–802
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with open access at Springerlink.com
0001-9054/15/030791-12
published online August 6, 2014
DOI 10.1007/s00010-014-0270-x

Aequationes Mathematicae

The characterization of some set-valued iteration semigroups

Grażyna Łydzińska

Abstract. We give the characterization of set-valued iteration semigroups which are the counterparts of the fundamental form of continuous iteration semigroups of single-valued functions on an interval.

Mathematics Subject Classification. 39B12, 39B52, 26E25, 26A18.

Keywords. Iteration theory, Set-valued function, Iteration semigroup, Set-valued iteration semigroup.

1. Introduction

Let X be an arbitrary set X. A multifunction $F : (0,\infty) \times X \to 2^X$ is said to be a *set-valued iteration semigroup* if

$$F^{s+t}(x) = F^t\big(F^s(x)\big) \quad \text{for } x \in X \text{ and } s,t \in (0,\infty).$$

$\big($We will write $F^t(x)$ instead of $F(t,x)$.$\big)$ This notion was introduced and investigated by Smajdor in [9] (see also e.g. [10]), studied by Olko (see e.g. [11]) and by Zdun in [13]. In [5] we introduced a family of set-valued functions which now will be denoted by (A) (see Sect. 2) and we showed $\big($see [5, Remarks 1 and 3]$\big)$ that F given by (A) is a set-valued counterpart of the fundamental form of iteration semigroups for single-valued functions which can be found in [2, Chap. IX, Sec. 1], [12, Theorems 5.1–8.1], [8, p. 98–99], [3, Chap. I, Sec. 1.7] $\big($cf. also [1, Theorem 1]$\big)$. In [7] we studied a lower semicontinuity of F given by (A).

The main aim of the present paper is to find the necessary and sufficient conditions under which F given by (A) is a set-valued iteration semigroup.

In [6] we introduced the definition of *expanding iteration semigroup* postulating that F satisfies the condition

$$F^t\big(F^s(x)\big) \subset F^{s+t}(x) \quad \text{for } x \in X \text{ and } s,t \in (0,\infty)$$

and in [4] we proved the following result, which will be useful in this paper.

Theorem 1. (see [4, Theorem]) *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A)*. If F is an expanding iteration semigroup then F is an iteration semigroup.*

2. Preliminaries

Fix a set X and a set-valued function $A : X \to 2^{\mathbb{R}}$ with non-empty values. Put

$$S := A(X) \quad \text{and} \quad q := \sup S.$$

Throughout this paper we will always assume that

(H) for every $s, t \in (0,\infty)$ and $x, z \in X$ with $\big[A(x) + s + t\big] \cap A(z) \neq \emptyset$ there exists $y \in X$ satisfying the conditions

$$[A(x) + s] \cap A(y) \neq \emptyset \tag{1}$$

and

$$[A(y) + t] \cap A(z) \neq \emptyset. \tag{2}$$

Notice that if S is an interval, then (H) holds $\big($see also [5, Proposition 1]$\big)$.

For every $x \in X$ define

$$\tau(x) := q - \inf A(x).$$

Fact 1. (see [5, Theorem 1 and Lemma 1]) *Let $t \in (0,\infty)$ and $x \in X$. If $t < \tau(x)$ then $[A(x) + t] \cap S \neq \emptyset$ and if $t > \tau(x)$ then $[A(x) + t] \cap S = \emptyset$.*

Fact 2. (see [5, Corollary 2]) *For every $x \in X$ we have*

$$\big[A(x) + \tau(x)\big] \cap S \subset \{q\}.$$

Let $e : (0,\infty) \times X \to [0,\infty)$ be defined by

$$e(t, x) := \min\{t, \tau(x)\}.$$

Now put

$$F^t(x) := A^-\big(A(x) + e(t,x)\big), \tag{A}$$

where

$$A^-(V) := \{x \in X : \ A(x) \cap V \neq \emptyset\}$$

for every $V \subset \mathbb{R}$.

Now we show the following easy remark.

Remark 1. Let $F : (0,\infty) \times X \to 2^X$ be given by (A). Assume that $x, y \in X$ and $t \in (0,\infty)$. If $[A(x) + t] \cap A(y) \neq \emptyset$ then $y \in F^t(x)$.

Proof. Assume that $[A(x) + t] \cap A(y) \neq \emptyset$. Then, by Fact 1, we have $t \leqslant \tau(x)$. Hence, due to (A), we get

$$y \in A^-\big(A(x) + t\big) = F^t(x).$$

□

Fact 3. (see [5, Lemma 3]) *Let* $F : (0,\infty) \times X \to 2^X$ *be given by* (A) *and let* $t \in (0,\infty)$ *and* $x \in X$*. If* $t < \tau(x)$ *then*

$$F^t(x) = A^-\big(A(x)+t\big) \neq \emptyset$$

and if $t \geqslant \tau(x)$ *then*

$$F^t(x) = \begin{cases} A^-(\{q\}), & \text{if } q \in S \text{ and } \inf A(x) \in A(x); \\ \emptyset & \text{otherwise.} \end{cases}$$

3. Iteration semigroups

In this section we give the characterization of iteration semigroups F which are given by formula (A). We do it in three possible and distinct cases:

(i) $q = \infty$ (see Theorem 2),
(ii) $q \notin S$ and $q \neq \infty$ (see Theorem 3),
(iii) $q \in S$ (see Theorem 4).

Consider the following condition:

(H1) for every $x, z \in X$ and $s, t \in (0,\infty)$ with $s + t \leqslant \tau(x)$ if (1) and (2) hold for $y \in X$ then

$$[A(x) + s + t] \cap A(z) \neq \emptyset. \tag{3}$$

Notice that if A is single-valued then (H1) holds (see also [6, Remark 1]).

Remark 2. Assume that (H1) holds and $q = \infty$. Then for every $x \in X$ either card $A(x) = 1$ or diam $A(x) = \infty$.

Proof. Fix $x \in X$ and assume that diam $A(x) < \infty$. Then $\inf A(x) > -\infty$ and $\sup A(x) < \infty$. Suppose that card $A(x) > 1$. Let $u, w \in A(x)$ and assume that $u < w$. Obviously $u < \frac{u+w}{2} < w$ thus, since $\inf A(x) > -\infty$, we can find $s \in (0,\infty)$ such that

$$w \in A(x) + s \quad \text{and} \quad \frac{u+w}{2} < \inf\,[A(x) + s].$$

Similarly, by the inequality $\sup A(x) < \infty$, there exists $t \in (0,\infty)$ that satisfies the conditions

$$u \in A(x) - t \quad \text{and} \quad \sup\,[A(x) - t] < \frac{u+w}{2}.$$

Therefore, obviously,

$$[A(x) + s] \cap A(x) \neq \emptyset \quad \text{and} \quad [A(x) - t] \cap A(x) \neq \emptyset$$

and $s + t < \tau(x) = \infty$. On the other hand, by the following inequality

$$\sup\,[A(x) - t] < \inf\,[A(x) + s],$$

we have

$$[A(x)+s] \cap [A(x)-t] = \emptyset,$$

which contradicts condition (H1). □

In [6] we proved the following three facts.

Fact 4. (see [6, Proposition 1]) *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A). *If F is an expanding iteration semigroup then* (H1) *holds.*

Fact 5. (see [6, Lemma 1]) *Assume* (H1) *and let $F : (0,\infty) \times X \to 2^X$ be given by* (A). *If $x \in X$ and* $\operatorname{card} A(x) = 1$ *then*

$$F^t\big(F^s(x)\big) \subset F^{s+t}(x)$$

holds for every $s, t \in (0,\infty)$ such that $s + t \leqslant \tau(x)$.

Fact 6. (see [6, Corollary 1]) *Assume* (H1) *and let $F : (0,\infty) \times X \to 2^X$ be given by* (A). *If either*

(i) $q = \infty$,

or

(ii) $\inf A(x) = -\infty$ *for every $x \in X$,*

then F is an iteration semigroup.

Notice that the next simple result, which follows immediately from Facts 4 and 6, gives the necessary and sufficient condition under which a multifunction given by (A) is an iteration semigroup in the case when $q = \infty$.

Theorem 2. *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A). *Assume that $q = \infty$. Then F is an iteration semigroup if and only if* (H1) *is satisfied.*

Now we deal with the case when $q \notin S$ and $q \neq \infty$. In this order we quote the following two facts which were shown in [6].

Fact 7. (see [6, Theorem 4]) *Assume that* (H1) *holds, $q \notin S$ and*

$$\operatorname{card} A(x) = 1 \quad \text{or} \quad \inf A(x) = -\infty \qquad \text{for} \quad x \in X. \tag{4}$$

Assume also that for every $x, y \in X$ if

$$\operatorname{card} A(x) = 1 \quad \text{and} \quad \inf A(y) = -\infty$$

then

$$\sup A(y) \leqslant u \qquad \text{where} \quad \{u\} = A(x).$$

Then (A) *defines an iteration semigroup $F : (0,\infty) \times X \to 2^X$.*

Fact 8. (see [6, Proposition 2]) *Let $F : (0,\infty) \times X \to 2^X$ be given by* (A). *Assume that F is an iteration semigroup, $q \notin S$ and $q \neq \infty$. If $x, y \in X$ and* $\inf A(y) = -\infty < \inf A(x)$ *then* $\sup A(y) \leqslant \inf A(x)$.

We also make use of the following result.

Lemma 1. (see [4, Lemma]) *Let* $F : (0, \infty) \times X \to 2^X$ *given by* (A) *be an expanding iteration semigroup and let* $x \in X$ *be such that* $\operatorname{card} A(x) > 1$ *and* $\tau(x) < \infty$. *Then* $\inf A(x) \in A(x)$ *and* $q \in A(x)$. *Moreover, if* $y \in X$ *then either* $\sup A(y) \leqslant \inf A(x)$ *or* $q \in A(y)$.

Corollary 1. *Let* $F : (0, \infty) \times X \to 2^X$ *given by* (A) *be an expanding iteration semigroup. Assume that* $q \notin S$. *Then for every* $x \in X$ *either* $\operatorname{card} A(x) = 1$ *or* $\operatorname{diam} A(x) = \infty$.

In particular, if moreover $q \neq \infty$ *then* (4) *holds.*

Proof. In the case $q = \infty$ it is enough to make use of Fact 4 and Remark 2.

Assume that $q \neq \infty$. Since $q \notin S$, by Lemma 1, for every $x \in X$ either $\operatorname{card} A(x) = 1$ or $\tau(x) = \infty$. If $x \in X$ and $\tau(x) = \infty$ then, due to the definition of $\tau(x)$ and the assumption $q \neq \infty$, we have $\inf A(x) = -\infty$. □

The next theorem follows immediately from Fact 4, Corollary 1, Fact 8 and Fact 7, and gives the necessary and sufficient condition under which F given by (A) is an iteration semigroup in the case when $q \notin S$ and $q \neq \infty$.

Theorem 3. *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A). *Assume that* $q \notin S$ *and* $q \neq \infty$. *Then the following conditions are equivalent:*

(i) F *is an iteration semigroup;*
(ii) *conditions* (H1) *and* (4) *are fulfilled and for every* $x, y \in X$ *if*

$$\operatorname{card} A(x) = 1 \quad \textit{and} \quad \inf A(y) = -\infty$$

then

$$\sup A(y) \leqslant u \qquad \textit{where} \quad \{u\} = A(x).$$

Now we pass to the case when $q \in S$.

Define the following sets:

$$\begin{aligned} \mathcal{L} &:= \big\{A(x)\colon \ x \in X, \ \inf A(x) = -\infty \ \text{ and } \ q \notin A(x)\big\}, \\ \mathcal{S} &:= \big\{A(x)\colon \ x \in X, \ \operatorname{card} A(x) = 1 \ \text{ and } \ A(x) \neq \{q\}\big\}, \\ \mathcal{P}_{-\infty} &:= \big\{A(x)\colon \ x \in X, \ \inf A(x) = -\infty \ \text{ and } \ q \in A(x)\big\}, \\ \mathcal{P} &:= \big\{A(x)\colon \ x \in X, \ \inf A(x) \in A(x) \ \text{ and } \ q \in A(x)\big\}. \end{aligned}$$

Observe that the above sets are pairwise disjoint.

Remark 3. Let $F : (0, \infty) \times X \to 2^X$ be given by (A). Assume that F is an iteration semigroup and $q \in S$. Then

$$A(x) \in \mathcal{L} \cup \mathcal{S} \cup \mathcal{P}_{-\infty} \cup \mathcal{P} \quad \text{for } \ x \in X.$$

Proof. Fix $x \in X$. Of course if $\operatorname{card} A(x) = 1$ then $A(x) \in \mathcal{S} \cup \mathcal{P}$. Assume that $\operatorname{card} A(x) > 1$. If $\tau(x) = \infty$ then since $q \in S$ we get $\inf A(x) = -\infty$ and consequently $A(x) \in \mathcal{L} \cup \mathcal{P}_{-\infty}$. If $\tau(x) < \infty$ then, by Lemma 1, we obtain that $A(x) \in \mathcal{P}$. □

Let $\mathcal{A}$ and $\mathcal{B}$ be arbitrary families of subsets of $\mathbb{R}$. We will write $\mathcal{A} \preceq \mathcal{B}$ if

$$\sup A \leqslant \inf B$$

for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

Proposition 1. *Let $F : (0, \infty) \times X \to 2^X$ be given by* (A)*. Assume that F is an iteration semigroup and $q \in S$. Then each of the following conditions holds:*

(i) $\mathcal{L} \preceq \mathcal{S} \cup \mathcal{P}$;
(ii) $\mathcal{S} \preceq \mathcal{P}$;
(iii) $\mathcal{S} = \emptyset$ *or* $\mathcal{P}_{-\infty} = \emptyset$;
(iv) $\mathcal{L} = \emptyset$ *or* $\mathcal{P}_{-\infty} = \emptyset$ *or* $\mathcal{P} = \emptyset$;
(v) *for every $x, y \in X$ if $A(x) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$, $A(y) \in \mathcal{P}$, $\inf A(x) < \inf A(y)$ and there exists $s \in \big(0, \inf A(y) - \inf A(x)\big)$ satisfying* (1)*, then for every $P \in \mathcal{P} \cup \mathcal{P}_{-\infty}$ and $t \in \big[\tau(y), \tau(x) - s\big)$*

$$\big[A(x) + s + t\big] \cap P \neq \emptyset; \tag{5}$$

(vi) *for every $x, y \in X$ if $A(x) \in \mathcal{L}$, $A(y) \in \mathcal{S} \cup \mathcal{P}$ and if $s \in (0, \infty)$ satisfies condition* (1)*, then* (5) *holds for every $P \in \mathcal{P}$ and $t \geqslant \tau(y)$.*

Proof. At first we prove (i). Suppose that (i) does not hold. Thus there exist $x, y \in X$ such that $A(x) \in \mathcal{S} \cup \mathcal{P}$, $A(y) \in \mathcal{L}$ and

$$\inf A(x) < \sup A(y).$$

We can find $s \in (0, \infty)$ satisfying $\big[A(x)+s\big] \cap A(y) \neq \emptyset$. Hence, by Remark 1, we have $y \in F^s(x)$. Since $\inf A(y) = -\infty$, we get $\tau(y) = \infty$. Obviously $\tau(x) < \infty$. Fact 1 implies that $s \leqslant \tau(x)$. Take

$$t > \tau(x) - s \qquad (t < \infty) \tag{6}$$

such that

$$\big[A(y) + t\big] \cap A(y) \neq \emptyset.$$

Of course, by Remark 1, $y \in F^t(y)$. We have shown that $y \in F^t\big(F^s(x)\big)$. On the other hand, on account of (6), Fact 3 and the definition of $\mathcal{S}$ and $\mathcal{P}$,

$$F^{s+t}(x) = A^-\big(\{q\}\big).$$

Since F is an iteration semigroup, we have

$$y \in F^t\big(F^s(x)\big) = F^{s+t}(x) = A^-\big(\{q\}\big),$$

whence $q \in A(y)$, which contradicts the definition of $\mathcal{L}$.

The condition (ii) follows immediately from the second part of Lemma 1.

Now pass to the proof of (iii). Suppose that there exist points $x, y \in X$ such that $A(x) \in \mathcal{S}$ and $A(y) \in \mathcal{P}_{-\infty}$. Of course $\tau(x) \in (0, \infty)$ and $q \in A(x) + \tau(x)$. Thus

$$\big[A(x) + \tau(x)\big] \cap A(y) \neq \emptyset.$$

Therefore by Remark 1

$$y \in F^{\tau(x)}(x). \tag{7}$$

Let $t \in (0, \infty)$ satisfy the condition

$$\big[A(y) + t\big] \cap A(x) \neq \emptyset.$$

By Remark 1, we get

$$x \in A^{-}\big(A(y) + t\big) = F^{t}(y). \tag{8}$$

According to (8), (7), the assumption on F and Fact 3, we have

$$x \in F^{t}(y) \subset F^{t}\big(F^{\tau(x)}(x)\big) = F^{\tau(x)+t}(x) = A^{-}\big(\{q\}\big),$$

which contradicts the definition of $\mathcal{S}$ and completes the proof of (iii).

(iv) Suppose that there exist $x, y, z \in X$ such that $A(x) \in \mathcal{P}$, $A(y) \in \mathcal{P}_{-\infty}$ and $A(z) \in \mathcal{L}$. Of course $\tau(x) < \infty$. Take

$$s \in \big(\tau(x), \infty\big). \tag{9}$$

Then, by Fact 3, we get $F^{s}(x) = A^{-}\big(\{q\}\big)$. Thus

$$y \in F^{s}(x). \tag{10}$$

Since $\inf A(y) = -\infty$ we can find $t \in (0, \infty)$ such that

$$\big[A(y) + t\big] \cap A(z) \neq \emptyset.$$

Therefore, due to Fact 3 and conditions (10) and (9), we have

$$z \in F^{t}(y) \subset F^{t}\big(F^{s}(x)\big) = F^{s+t}(x) = A^{-}\big(\{q\}\big),$$

which contradicts the definition of $\mathcal{L}$.

To show condition (v) take $x, y \in X$ and assume that $A(x) \in \mathcal{P} \cup \mathcal{P}_{-\infty}$, $A(y) \in \mathcal{P}$ and $\inf A(x) < \inf A(y)$. Moreover assume that $s \in \big(0, \inf A(y) - \inf A(x)\big)$ satisfies condition (1). Notice that, by Remark 1, $y \in F^{s}(x)$. Moreover

$$s < \inf A(y) - \inf A(x) = \tau(x) - \tau(y) \leqslant \tau(x),$$

whence the interval $\big[\tau(y), \tau(x) - s\big)$ is non-empty. Take an arbitrary number $t \in \big[\tau(y), \tau(x) - s\big)$ and a set $P = A(z) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$, where $z \in X$. Due to Fact 3 we get $F^{t}(y) = A^{-}\big(\{q\}\big)$. Hence

$$z \in A^{-}\big(\{q\}\big) \subset F^{t}\big(F^{s}(x)\big) = F^{s+t}(x). \tag{11}$$

Observe that, by the choice of t, we have $s + t < \tau(x)$. Thus, by (11) and Fact 3, we obtain

$$\big[A(x) + s + t\big] \cap P \neq \emptyset,$$

which completes the proof of (v).

Pass to (vi). Take $x, y \in X$ such that $A(x) \in \mathcal{L}$ and $A(y) \in \mathcal{S} \cup \mathcal{P}$. Let $s \in (0, \infty)$ be an arbitrary number satisfying condition (1). Since $A(y) \in \mathcal{S} \cup \mathcal{P}$, we have $\tau(y) < \infty$. Fix $t \in \big[\tau(y), \infty\big)$. If $t = 0$ then of course $\tau(y) = 0$, whence

$A(y)=\{q\}$. Therefore, by (1), we have $q\in A(x)+s$ and (5) holds. Consider the case when $t\neq 0$. Then $F^t(y)=A^-\big(\{q\}\big)$. Since $A(x)\in\mathcal{L}$, we get $\tau(x)=\infty$. Hence, by (1) and Fact 3, we obtain $y\in F^s(x)$. F is an iteration semigroup, thus, due to Fact 3,

$$A^-\big(\{q\}\big)\subset F^t\big(F^s(x)\big)=F^{s+t}(x)=A^-\big(A(x)+s+t\big).$$

In particular the condition

$$\big[A(x)+s+t\big]\cap P\neq\emptyset$$

holds for every set $P\in\mathcal{P}$. □

Theorem 4. *Let $F:(0,\infty)\times X\to 2^X$ be given by* (A). *Assume that $q\in S$. Then F is an iteration semigroup if and only if condition* (H1) *and each of the following conditions hold:*

(a) $A(x)\in\mathcal{L}\cup\mathcal{S}\cup\mathcal{P}_{-\infty}\cup\mathcal{P}$ *for every $x\in X$;*
(b) $\mathcal{L}\preceq\mathcal{S}\cup\mathcal{P}$*;*
(c) $\mathcal{S}\preceq\mathcal{P}$*;*
(d) $\mathcal{S}=\emptyset$ *or* $\mathcal{P}_{-\infty}=\emptyset$*;*
(e) $\mathcal{L}=\emptyset$ *or* $\mathcal{P}_{-\infty}=\emptyset$ *or* $\mathcal{P}=\emptyset$*;*
(f) *for every $x,y\in X$ if $A(x)\in\mathcal{P}_{-\infty}\cup\mathcal{P}$, $A(y)\in\mathcal{P}$, $\inf A(x)<\inf A(y)$ and there exists $s\in\big(0,\inf A(y)-\inf A(x)\big)$ satisfying* (1)*, then for every $P\in\mathcal{P}\cup\mathcal{P}_{-\infty}$ and $t\in\big[\tau(y),\tau(x)-s\big)$ condition* (5) *holds;*
(g) *for every $x,y\in X$ if $A(x)\in\mathcal{L}$, $A(y)\in\mathcal{S}\cup\mathcal{P}$ and $s\in(0,\infty)$ satisfies* (1)*, then condition* (5) *holds for every $P\in\mathcal{P}$ and $t\geqslant\tau(y)$.*

Proof. If F is an iteration semigroup then the demanded conditions follow immediately from Fact 4, Remark 3 and Proposition 1.

Pass to the proof of the converse implication. Assume that conditions (H1) and (a)–(g) hold. According to Theorem 1 it is enough to show that F is an expanding iteration semigroup. Fix $x\in X$ and $s,t\in(0,\infty)$. Take $z\in F^t\big(F^s(x)\big)$ and let $y\in F^s(x)$ be such that $z\in F^t(y)$. We divide this proof into four parts, depending on the form of $A(x)$ $\big($see (a)$\big)$.

Part 1. Assume that $A(x)\in\mathcal{L}$. Of course $\tau(x)=\infty$. Hence, by Fact 3, condition (1) holds. Additionally $A(y)\in\mathcal{L}\cup\mathcal{S}\cup\mathcal{P}_{-\infty}\cup\mathcal{P}$ (see condition (a)).

If $t<\tau(y)$ then, according to Fact 3, we get that $z\in A^-\big(A(y)+t\big)$, so

$$\big[A(y)+t\big]\cap A(z)\neq\emptyset.$$

Therefore, by (1) and (H1), we have

$$\big[A(x)+s+t\big]\cap A(z)\neq\emptyset,$$

hence, on account of Remark 1, we get $z\in F^{s+t}(x)$.

Thus, assume that $t\geqslant\tau(y)$. Then $\tau(y)<\infty$ and hence by (a), $A(y)\in\mathcal{S}\cup\mathcal{P}$. Moreover, due to Fact 3

$$z\in F^t(y)=A^-\big(\{q\}\big). \tag{12}$$

Then, from conditions (12), (a), (d) and (e), we obtain $A(z) \in \mathcal{P}$. Hence, by (1) and (g) we have

$$\big[A(x) + s + t\big] \cap A(z) \neq \emptyset.$$

Thus, according to Remark 1, we get $z \in F^{s+t}(x)$.

Part 2. Assume that $A(x) \in \mathcal{P}_{-\infty}$. Obviously $\tau(x) = \infty$, so $s < \tau(x)$ and, by Fact 3,

$$\big[A(x) + s\big] \cap A(y) \neq \emptyset.$$

Due to conditions (a) and (d) we have

$$A(y) \in \mathcal{L} \cup \mathcal{P}_{-\infty} \cup \mathcal{P}. \tag{13}$$

If $t < \tau(y)$ then

$$z \in F^t(y) = A^-\big(A(y) + t\big),$$

whence $\big[A(y) + t\big] \cap A(z) \neq \emptyset$. According to (H1) we infer that

$$\big[A(x) + s + t\big] \cap A(z) \neq \emptyset$$

and, by Remark 1, we get $z \in F^{s+t}(x)$.

Now assume that $t \geqslant \tau(y)$. Then $\tau(y) < \infty$ and, by (13), $A(y) \in \mathcal{P}$. Hence, an account of Fact 3,

$$z \in F^t(y) = A^-\big(\{q\}\big).$$

Due to (a) we have $A(z) \in \mathcal{P} \cup \mathcal{P}_{-\infty}$. Thus, by (f),

$$\big[A(x) + s + t\big] \cap A(z) \neq \emptyset,$$

whence, by Remark 1, $z \in F^{s+t}(x)$.

Part 3. Assume that $A(x) \in \mathcal{S}$. If $s + t \leqslant \tau(x)$ then, according to Fact 5, the inclusion interesting for us holds.

Consider the case when $s + t > \tau(x)$. Then, by Fact 3,

$$F^{s+t}(x) = A^-\big(\{q\}\big). \tag{14}$$

If $t \geqslant \tau(y)$ then

$$z \in F^t(y) = A^-\big(\{q\}\big) = F^{s+t}(x).$$

So assume that $t < \tau(y)$. According to Fact 3 we obtain (2).

We will show that

$$A(y) \in \mathcal{P}. \tag{15}$$

Consider the case when $s < \tau(x)$. Then condition $\big[A(x) + s\big] \cap A(y) \neq \emptyset$ is satisfied. Hence, since $A(x) \in \mathcal{S}$ and by (b), we obtain that $A(y) \notin \mathcal{L}$. Moreover, condition (d) implies that $A(y) \notin \mathcal{P}_{-\infty}$. Consequently, by (a), we get $A(y) \in \mathcal{S} \cup \mathcal{P}$. Suppose that $A(y) \in \mathcal{S}$. Then, by (1), we get $A(y) = A(x) + s$, whence, by (2), we obtain $\big[A(x) + s\big] \cap \big[A(z) - t\big] \neq \emptyset$. Due to Fact 1, we have

$s+t \leqslant \tau(x)$ which contradicts the assumption on $s+t$. Hence (15) holds. Now assume $s \geqslant \tau(x)$. Then, according to Fact 3,

$$y \in F^s(x) = A^-\big(\{q\}\big),$$

so, by (a) and (d) we also obtain (15).

Thus, an account of (2), (15), (a)–(d), we have $A(z) \in \mathcal{P}$, whence, by the equality (14),

$$z \in A^-\big(\{q\}\big) = F^{s+t}(x).$$

Part 4. Assume that $A(x) \in \mathcal{P}$. At first consider the case when

$$s+t < \tau(x). \tag{16}$$

Of course $s < \tau(x)$ and, by Fact 3, condition (1) holds. If $t < \tau(y)$ then

$$\big[A(y)+t\big] \cap A(z) \neq \emptyset$$

and, due to (H1), we get $z \in F^{s+t}(x)$. So assume that

$$t \geqslant \tau(y). \tag{17}$$

Hence, by Fact 3 we have $z \in A^-\big(\{q\}\big)$ and in particular, by (a), $A(z) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$. By the inequality (17) we obtain $\tau(y) < \infty$. Thus $A(y) \notin \mathcal{L}$ and $A(y) \notin \mathcal{P}_{-\infty}$. Hence, according to (1), (a) and (c), we infer that $A(y) \in \mathcal{P}$. By (16) and (17) we have

$$s < \tau(x) - \tau(y) = \inf A(y) - \inf A(x).$$

An account of (f) we get

$$\big[A(x)+s+t\big] \cap A(z) \neq \emptyset,$$

hence, by Remark 1, $z \in F^{s+t}(x)$.

Now consider the case when $s+t \geqslant \tau(x)$. According to Fact 3 we have

$$F^{s+t}(x) = A^-\big(\{q\}\big).$$

If $t \geqslant \tau(y)$ then

$$z \in F^t(y) = A^-\big(\{q\}\big) = F^{s+t}(x).$$

So assume that $t < \tau(y)$. Therefore, by Fact 3 we get (2). If $s < \tau(x)$ then, again by Fact 3, we have $\big[A(x)+s\big] \cap A(y) \neq \emptyset$. Thus, due to (a), (b) and (c) we infer that $A(y) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$. Notice that, in the case when $s \geqslant \tau(x)$, we have

$$y \in F^s(x) = A^-\big(\{q\}\big).$$

Hence also in this case $A(y) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$. Observe that $A(z) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$. In fact, if $A(y) \in \mathcal{P}_{-\infty}$ then, by (d), $A(z) \notin \mathcal{S}$. On the other hand in this case condition (e) implies that $A(z) \notin \mathcal{L}$. In the case $A(y) \in \mathcal{P}$, by (2) and (a)-(c), we also obtain that $A(z) \in \mathcal{P}_{-\infty} \cup \mathcal{P}$. Now therefore

$$z \in A^-\big(\{q\}\big) = F^{s+t}(x).$$

$\square$

As a conclusion we obtain the following theorem.

Theorem 5. *Let* $F : (0, \infty) \times X \to 2^X$ *be given by* (A). *Then* F *is an iteration semigroup if and only if* (H1) *holds and either*

(i) $q = \infty$,

or

(ii) $q \notin S$ *and* $q \neq \infty$,

$$A(x) \in \mathcal{L} \cup \mathcal{S} \quad \textit{for } \ x \in X$$

and $\mathcal{L} \preceq \mathcal{S}$,

or

(iii) $q \in S$ *and conditions* (a)–(g) *are fulfilled.*

Acknowledgements

This research was supported by the Department of Mathematics, University of Silesia, Katowice, Poland (Iterative Functional Equations and Real Analysis Program).

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Received: November 3, 2013
Revised: April 28, 2014